

The Generalized Hodge conjecture for 1-cycles and codimension two algebraic cycles

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Abstract

In this paper, we prove that the statement: “The (Generalized) Hodge Conjecture holds for codimension-two cycles on a smooth projective variety X ” is a birationally invariant statement, that is, if the statement is true for X , it is also true for all smooth varieties X' which are birationally equivalent to X . We also prove the analogous result for 1-cycles. As direct corollaries, the Hodge Conjecture holds for smooth rational projective manifolds with dimension less than or equal to five, and, the Generalized Hodge Conjecture holds for smooth rational projective manifolds with dimension less than or equal to four.

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1 Introduction

In this paper, all varieties are defined over \mathbb{C} . Let X be a smooth projective variety with dimension n . Let $\mathcal{Z}_p(X)$ be the space of algebraic p -cycles on X . Set $\mathcal{Z}^{n-p}(X) \equiv \mathcal{Z}_p(X)$. There is a natural map

$$cl_q : \mathcal{Z}^q(X) \rightarrow H^{2q}(X, \mathbb{Z})$$

called **the cycle class map**.

Tensoring with \mathbb{Q} , we have

$$cl_q \otimes \mathbb{Q} : \mathcal{Z}^q(X) \otimes \mathbb{Q} \rightarrow H^{2q}(X, \mathbb{Q}).$$

It is well known that $cl_q(\mathcal{Z}_q(X)) \subseteq H^{q,q}(X) \cap \rho(H^{2q}(X, \mathbb{Z}))$, where $\rho : H^{2q}(X, \mathbb{Z}) \rightarrow H^{2q}(X, \mathbb{C})$ is the coefficient homomorphism and $H^{q,q}(X)$ denotes the (q, q) -component in the Hodge decomposition (cf. [GH], [Lew1]). There are known examples where $cl_q(\mathcal{Z}_q(X)) \neq H^{q,q}(X) \cap \rho(H^{2q}(X, \mathbb{Z}))$ (cf. [BCC] p.134-125, [Lew2]). We recall:

The Hodge Conjecture (for codimension- q cycles): *The rational cycle class map*

$$cl_q \otimes \mathbb{Q} : \mathcal{Z}^q(X) \otimes \mathbb{Q} \rightarrow H^{q,q}(X) \cap H^{2q}(X, \mathbb{Q})$$

is surjective.

The Hodge Conjecture over \mathbb{Z} : *The rational cycle class map*

$$cl_q : \mathcal{Z}^q(X) \rightarrow H^{q,q}(X) \cap \rho(H^{2q}(X, \mathbb{Z}))$$

is surjective.

We shall denote by $\text{Hodge}^{q,q}(X, \mathbb{Q})$ the statement that: “The Hodge Conjecture for codimension- q cycles is true for X ”. Similarly, we denote by $\text{Hodge}^{q,q}(X, \mathbb{Z})$ the corresponding statement for the Hodge Conjecture over \mathbb{Z} .

More generally, we can define a filtration on $H_k(X, \mathbb{Q})$ as follows:

Definition 1.1 ([FM], §7) *Denote by $\tilde{F}_p H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$ the maximal sub-(Mixed) Hodge structure of span $k - 2p$. (See [Gro] and [FM].) The*

sub- \mathbb{Q} vector spaces $\tilde{F}_p H_k(X, \mathbb{Q})$ form a decreasing filtration of sub-Hodge structures:

$$\cdots \subseteq \tilde{F}_p H_k(X, \mathbb{Q}) \subseteq \tilde{F}_{p-1} H_k(X, \mathbb{Q}) \subseteq \cdots \subseteq \tilde{F}_0 H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$$

and $\tilde{F}_p H_k(X, \mathbb{Q})$ vanishes if $2p > k$. This filtration is called the **Hodge filtration**.

A homological version of the arithmetic filtration (see [[Lew1], §7]) is given in the following definition:

Definition 1.2 ([FM], §7) Denote by $G_p H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q})$ the \mathbb{Q} -vector subspace of $H_k(X, \mathbb{Q})$ generated by the images of mappings $H_k(Y, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q})$, induced from all morphisms $Y \rightarrow X$ of varieties of dimension $\leq k - p$. The subspaces $G_p H_k(X, \mathbb{Q})$ also form a decreasing filtration called the **geometric filtration**:

$$\cdots \subseteq G_p H_k(X, \mathbb{Q}) \subseteq G_{p-1} H_k(X, \mathbb{Q}) \subseteq \cdots \subseteq G_0 H_k(X, \mathbb{Q}) \subseteq H_k(X, \mathbb{Q}).$$

Since X is smooth, the Weak Lefschetz Theorem implies that $G_0 H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Q})$. Since $H_k(Y, \mathbb{Q})$ vanishes for k greater than twice the dimension of Y , $G_p H_k(X, \mathbb{Q})$ vanishes if $2p > k$.

It was proved in [Gro] that, for any smooth variety X , the geometric filtration is finer than the Hodge filtration, i.e., $G_p H_k(X, \mathbb{Q}) \subseteq \tilde{F}_p H_k(X, \mathbb{Q})$, for all p and k .

The Generalized Hodge Conjecture: For any smooth variety X ,

$$G_p H_k(X, \mathbb{Q}) = \tilde{F}_p H_k(X, \mathbb{Q}) \tag{1}$$

for all p and k . Using the notation given in [Lew1], we denote by $\widetilde{GHC}(p, k, X)$ the assertion that (1) is true.

Definition 1.3 The **Lawson homology** $L_p H_k(X)$ of p -cycles is defined by

$$L_p H_k(X) = \pi_{k-2p}(\mathcal{Z}_p(X)) \quad \text{for } k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [F], [L1]). For general background, the reader is referred to Lawson's survey paper [L2].

There are two special cases.

- (a) If $p = 0$, then for all $k \geq 0$, $L_0 H_k(X) \cong H_k(X, \mathbb{Z})$ by Dold-Thom Theorem [DT].
- (b) If $k = 2p$, then $L_p H_{2p}(X) = \mathcal{Z}_p(X) / \mathcal{Z}_p(X)_{alg}$, where $\mathcal{Z}_p(X)_{alg}$ denotes the algebraic p -cycles on X which are algebraic equivalent to zero.

In [FM], Friedlander and Mazur showed that there are natural maps, called **cycle class maps**

$$\Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X, \mathbb{Z}).$$

Define

$$L_p H_k(X)_{hom} := \ker \{ \Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X, \mathbb{Z}) \}.$$

$$T_p H_k(X) := \text{Im} \{ \Phi_{p,k} : L_p H_k(X) \rightarrow H_k(X, \mathbb{Z}) \}$$

and

$$T_p H_k(X, \mathbb{Q}) := T_p H_k(X) \otimes \mathbb{Q}.$$

It was proved in [[FM], §7] that, for any smooth variety X , $T_p H_k(X, \mathbb{Q}) \subseteq G_p H_k(X, \mathbb{Q})$ for all p and k . Hence

$$T_p H_k(X, \mathbb{Q}) \subseteq G_p H_k(X, \mathbb{Q}) \subseteq \tilde{F}_p H_k(X, \mathbb{Q}). \quad (2)$$

In this paper, we will use the tools in Lawson homology and the methods given in [H1] to show the following main result:

Theorem 1.1 *Let X be a smooth projective variety. If the Hodge conjecture for codimension 2 cycles over \mathbb{Z} holds for X , i.e., if we have $\text{Hodge}^{2,2}(X, \mathbb{Z})$, then it holds for any smooth projective variety X' birational to X . That is, $\text{Hodge}^{2,2}(X, \mathbb{Z})$ is a birationally invariant assertion for smooth varieties X .*

Remark 1.1 *The above theorem remains true if \mathbb{Z} is replaced by \mathbb{Q} . Since $\text{Hodge}^{2,2}(X, \mathbb{Q})$ implies $\text{Hodge}^{n-2,n-2}(X, \mathbb{Q})$ for $n \geq 4$ (cf. [[Lew1], p.91]), $\text{Hodge}^{n-2,n-2}(X, \mathbb{Q})$ is also a birationally invariant property of smooth n -dimensional varieties X .*

As a corollary, we have

Corollary 1.1 *If X is a rational manifold with $\dim(X) \leq 5$, then the Hodge conjecture $\text{Hodge}^{\text{p,p}}(X, \mathbb{Q})$ is true for $1 \leq p \leq \dim(X)$. In fact, $\text{Hodge}^{\text{p,p}}(X', \mathbb{Z})$ is true except possibly for $p = 3, \dim(X) = 5$.*

Remark 1.2 *By using the technique of the diagonal decomposition, Bloch and Srinivas [BS] showed that, for any smooth projective variety X , $\text{Hodge}^{2,2}(X, \mathbb{Q})$ holds if the Chow group of 0-cycles $\text{Ch}_0(X) \cong \mathbb{Z}$. Laterveer [Lat] generalized this technique and showed the Hodge Conjecture holds for a class of projective manifolds with small chow groups.*

Corollary 1.2 *Let X be a smooth projective variety of dimension ≤ 5 such that the Hodge Conjecture is known to be true, i.e., $\text{Hodge}^{\text{p,p}}(X, \mathbb{Q})$ holds for all p . Then the Hodge Conjecture holds for all smooth projective varieties X' which are birationally equivalent to X . Non-rational examples of such an X include general abelian varieties or the product of at most five elliptic curves. For more examples, the reader is referred to the survey book [Lew1].*

Our second main result is the following

Theorem 1.2 *The assertion $\widetilde{\text{GHC}}(n-2, k, X)$ is a birationally invariant property of smooth n -dimensional varieties X when $k \geq 2(n-2)$. More precisely, if $\widetilde{\text{GHC}}(n-2, k, X)$ holds for a smooth variety X , then $\widetilde{\text{GHC}}(n-2, k, X')$ holds for any smooth variety X' birational to X .*

We also show that

Proposition 1.1 *The assertion that “ $T_{n-2}H_k(X, \mathbb{Q}) = \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ holds” is a birationally invariant property of smooth n -dimensional varieties X when $k \geq 2(n-2)$.*

Similarly, for 1-cycles, we can show the following.

Proposition 1.2 *For integer $k \geq 2$, the assertion that “ $T_1H_k(X, \mathbb{Q}) = \tilde{F}_1H_k(X, \mathbb{Q})$ holds” is a birationally invariant property of smooth n -dimensional varieties X .*

and

Theorem 1.3 *For any integer $k \geq 2$, the assertion $\widetilde{\text{GHC}}(1, k, X)$ is a birationally invariant property of smooth varieties X .*

Remark 1.3 *For the case $k = \dim(X)$, Lewis has already obtained this result in [Lew1].*

Corollary 1.3 *For any smooth rational variety X with $\dim(X) \leq 4$, the Generalized Hodge Conjecture holds.*

The main tools used to prove this result are: the long exact localization sequence given by Lima-Filho in [Li], the explicit formula for Lawson homology of codimension-one cycles on a smooth projective manifold given by Friedlander in [F], and the weak factorization theorem proved by Włodarczyk in [W] and in [AKMW].

2 The Proof of the Main Theorems

Let X be a smooth projective manifold of dimension n . In the following, we will denote by $H_{p,q}(X)$ the image of $H^{n-p,n-q}(X)$ under the Poincare duality isomorphism $H^{2n-p-q}(X, \mathbb{C}) \cong H_{p+q}(X, \mathbb{C})$.

Let X be a smooth projective manifold and $i_0 : Y \hookrightarrow X$ be a smooth subvariety of codimension r . Let $\sigma : \tilde{X}_Y \rightarrow X$ be the blowup of X along Y , $i : D := \sigma^{-1}(Y) \hookrightarrow \tilde{X}_Y$ the exceptional divisor of the blowing up, and $\pi : D \rightarrow Y$ the restriction of σ to D . Set $U := X - Y \cong \tilde{X}_Y - D$. Denote by j_0 the inclusion $U \subset X$ and j the inclusion $U \subset \tilde{X}_Y$.

Now I list the Lemmas and Corollaries given in [H1].

Lemma 2.1 *For each p , we have the following commutative diagram*

$$\begin{array}{ccccccccc}
 \cdots \rightarrow & L_p H_k(D) & \xrightarrow{i_*} & L_p H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{\delta_*} & L_p H_{k-1}(D) & \rightarrow \cdots \\
 & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & \\
 \cdots \rightarrow & L_p H_k(Y) & \xrightarrow{(i_0)_*} & L_p H_k(X) & \xrightarrow{j_0^*} & L_p H_k(U) & \xrightarrow{(\delta_0)_*} & L_p H_{k-1}(Y) & \rightarrow \cdots
 \end{array}$$

Remark 2.1 *Since π_* is surjective (this follows from the explicit formula for the Lawson homology of D , i.e., the Projective Bundle Theorem in [FG]), it is easy to see that σ_* is surjective.*

Corollary 2.1 *If $p = 0$, then we have the following commutative diagram*

$$\begin{array}{ccccccccc}
 \cdots \rightarrow & H_k(D) & \xrightarrow{i_*} & H_k(\tilde{X}_Y) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{\delta_*} & H_{k-1}(D) & \rightarrow \cdots \\
 & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & \\
 \cdots \rightarrow & H_k(Y) & \xrightarrow{(i_0)^*} & H_k(X) & \xrightarrow{j_0^*} & H_k^{BM}(U) & \xrightarrow{(\delta_0)^*} & H_{k-1}(Y) & \rightarrow \cdots
 \end{array}$$

Moreover, if $x \in H_k(D)$ vanishes under π_* and i_* , then $x = 0 \in H_k(D)$.

Corollary 2.2 *If $p = n-2$, then we have the following commutative diagram*

$$\begin{array}{ccccccccc}
 \cdots \rightarrow & L_{n-2}H_k(D) & \xrightarrow{i_*} & L_{n-2}H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_{n-2}H_k(U) & \xrightarrow{\delta_*} & L_{n-2}H_{k-1}(D) & \rightarrow \cdots \\
 & \downarrow \pi_* & & \downarrow \sigma_* & & \downarrow \cong & & \downarrow \pi_* & \\
 \cdots \rightarrow & L_{n-2}H_k(Y) & \xrightarrow{(i_0)^*} & L_{n-2}H_k(X) & \xrightarrow{j_0^*} & L_{n-2}H_k(U) & \xrightarrow{(\delta_0)^*} & L_{n-2}H_{k-1}(Y) & \rightarrow \cdots
 \end{array}$$

Lemma 2.2 *For each p , we have the following commutative diagram*

$$\begin{array}{ccccccccc}
 \cdots \rightarrow & L_p H_k(D) & \xrightarrow{i_*} & L_p H_k(\tilde{X}_Y) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{\delta_*} & L_p H_{k-1}(D) & \rightarrow \cdots \\
 & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & \\
 \cdots \rightarrow & H_k(D) & \xrightarrow{i_*} & H_k(\tilde{X}_Y) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{\delta_*} & H_{k-1}(D) & \rightarrow \cdots
 \end{array}$$

In particular, it is true for $p = 1, n-2$.

Lemma 2.3 *For each p , we have the following commutative diagram*

$$\begin{array}{ccccccccc}
 \cdots \rightarrow & L_p H_k(Y) & \xrightarrow{(i_0)^*} & L_p H_k(X) & \xrightarrow{j^*} & L_p H_k(U) & \xrightarrow{(\delta_0)^*} & L_p H_{k-1}(Y) & \rightarrow \cdots \\
 & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k} & & \downarrow \Phi_{p,k-1} & \\
 \cdots \rightarrow & H_k(Y) & \xrightarrow{(i_0)^*} & H_k(X) & \xrightarrow{j^*} & H_k^{BM}(U) & \xrightarrow{(\delta_0)^*} & H_{k-1}(Y) & \rightarrow \cdots
 \end{array}$$

In particular, it is true for $p = 1, n-2$.

Remark 2.2 *All the commutative diagrams of long exact sequences remain commutative and exact after tensoring with \mathbb{Q} . We will use these Lemmas and corollaries with rational coefficients.*

The following result proved by Friedlander will be used several times:

Theorem 2.1 (Friedlander [F]) *Let X be any smooth projective variety of dimension n . Then we have the following isomorphisms*

$$\begin{cases} L_{n-1}H_{2n}(X) \cong \mathbb{Z}, \\ L_{n-1}H_{2n-1}(X) \cong H_{2n-1}(X, \mathbb{Z}), \\ L_{n-1}H_{2n-2}(X) \cong H_{n-1, n-1}(X, \mathbb{Z}) = NS(X) \\ L_{n-1}H_k(X) = 0 \quad \text{for } k > 2n. \end{cases}$$

where $NS(X)$ is the Néron-Severi group of X .

2.1 The Proof of Theorem 1.1 for a Blowup

In what follows we drop reference to the coefficient homomorphism ρ , and denote by $H_k(X, \mathbb{Z})$ its image in $H_k(X, \mathbb{C})$.

There are two cases to consider:

Case 1: If $\Phi_{n-2, 2(n-2)} : L_{n-2}H_{2(n-2)}(X) \rightarrow H_{2(n-2)}(X, \mathbb{Z}) \cap H_{n-2, n-2}(X)$ is surjective, we will show that $\Phi_{n-2, 2(n-2)} : L_{n-2}H_{2(n-2)}(\tilde{X}_Y) \rightarrow H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z}) \cap H_{n-2, n-2}(\tilde{X}_Y)$ is also surjective.

Let $b \in H_{n-2, n-2}(\tilde{X}_Y) \cap H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z})$. Set $a \equiv \sigma_*(b) \in H_{2(n-2)}(X, \mathbb{Z})$. Since σ_* preserves the type, we have $a \in H_{2(n-2)}(X, \mathbb{Z}) \cap H_{n-2, n-2}(X)$. Now by assumption, there exists an element $\tilde{a} \in L_{n-2}H_{2(n-2)}(X)$ such that $\Phi_{n-2, 2(n-2)}(\tilde{a}) = a$. Now since $L_{n-2}H_{2(n-2)}(\tilde{X}_Y) \rightarrow L_{n-2}H_{2(n-2)}(X)$ is surjective, there exists an element $\tilde{b} \in L_{n-2}H_{2(n-2)}(\tilde{X}_Y)$ such that $\sigma_*(\tilde{b}) = \tilde{a}$. Now $\Phi_{n-2, 2(n-2)}(\tilde{b}) - b$ is mapped to zero under σ_* on $H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z})$. By the commutative diagram in the long exact sequences in Corollary 2.1, there exists an element $c \in H_{2(n-2)}(D, \mathbb{Z})$ such that $i_*(c) = \Phi_{n-2, 2(n-2)}(\tilde{b}) - b$. Using Corollary 2.1 once again, we have $\pi_*(c) = 0$. This follows from the fact that $\dim(Y) = n - r \leq n - 2$ and hence $(i_0)_* : H_{2(n-2)}(Y, \mathbb{Z}) \rightarrow H_{2(n-2)}(X, \mathbb{Z})$ is injective. From the blowup formula for the singular homology, $i_*|_{\ker \pi_*}$ is injective. Now by assumption, b and \tilde{b} are non-torsion elements. Hence c is not a torsion element in $H_{2(n-2)}(D, \mathbb{Z})$, i.e., $c \in H_{2(n-2)}(D, \mathbb{Z})_{\text{free}}$, the torsion free part of $H_{2(n-2)}(D, \mathbb{Z})$.

Since i_* preserves the type, we have the following

Claim: $c \in H_{2(n-2)}(D, \mathbb{Z}) \cap H_{n-2, n-2}(D)$.

Proof. Since $H_{2(n-2)}(D, \mathbb{Z})_{\text{free}} \subset H_{2(n-2)}(D, \mathbb{C}) = H_{n-2, n-2}(D) \oplus H_{n-1, n-3}(D) \oplus H_{n-3, n-1}(D)$. Now $c = c_0 + c_1 + \bar{c}_1 \in H_{2(n-2)}(D, \mathbb{C})$ such that $c_0 \in H_{n-2, n-2}(D)$,

$c_1 \in H_{n-1,n-3}(D)$ and hence $\bar{c}_1 \in H_{n-3,n-1}(D)$. Note that the complexification of i_* is the map $i_* \otimes \mathbb{C} : H_{2(n-2)}(D, \mathbb{C}) \rightarrow H_{2(n-2)}(\tilde{X}_Y, \mathbb{C})$. If $i_* \otimes \mathbb{C}(c_1) = 0$, we have $c_1 = 0$. In fact, $i_* \otimes \mathbb{C}(c_1) = 0$ and the exactness of the long exact sequence in the upper row in Corollary 2.1 implies that an element $d \in H_{2(n-2)+1}^{BM}(U, \mathbb{C})$ such that $\delta_*(d) = c_1$. We use the commutative diagram in Corollary 2.1 again. From the commutativity of the diagram in Corollary 2.1, we have the image of d under the boundary map $(\delta_0)_*$ must zero in $H_{2(n-2)}(Y, \mathbb{C})$. This follows from the fact that the complex dimension of $\dim(Y) \leq n-2$ and the Hodge type of d is of type $(n-1, n-3)$. Now by the exactness of the long exact sequence in the lower row in Corollary 2.1, there exists an element $e \in H_{2(n-2)+1}(X, \mathbb{C})$ such that $j_0^*(e) = d$. It is well-known that $\sigma_* : H_{2(n-2)+1}(\tilde{X}_Y, \mathbb{C}) \rightarrow H_{2(n-2)+1}(X, \mathbb{C})$ is surjective. Therefore, there exists $\tilde{e} \in H_{2(n-2)+1}(\tilde{X}_Y, \mathbb{C})$ such that $\sigma_*(\tilde{e}) = e$. We get $d = j^*(\tilde{e})$ and hence $c_1 = 0 \in H_{2(n-2)}(D, \mathbb{C})$ by the exactness of the the upper row sequence in Corollary 2.1. This implies $\bar{c}_1 = 0$ and hence $c \in H_{n-2,n-2}(D)$. This finishes the proof of the claim. \square

Since $\dim D = n-1$, hence by Theorem 2.1, the map $\Phi_{n-2,2(n-2)} : L_{n-2}H_{2(n-2)}(D) \rightarrow H_{2(n-2)}(D, \mathbb{Z}) \cap H_{n-2,n-2}(D)$ is an isomorphism. Set $\tilde{c} \equiv \Phi_{n-2,2(n-2)}(c)$. Therefore, $\Phi_{n-2,2(n-2)}\{\tilde{b} - i_*(\tilde{c})\} = b$. Hence $\Phi_{n-2,2(n-2)} : L_{n-2}H_{2(n-2)}(\tilde{X}_Y) \rightarrow H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z}) \cap H_{n-2,n-2}(\tilde{X}_Y)$ is surjective.

On the other hand, we need to show

Case 2: If $\Phi_{n-2,2(n-2)} : L_{n-2}H_{2(n-2)}(\tilde{X}_Y) \rightarrow H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z}) \cap H_{n-2,n-2}(\tilde{X}_Y)$ is surjective, then $\Phi_{n-2,2(n-2)} : L_{n-2}H_{2(n-2)}(X) \rightarrow H_{2(n-2)}(X, \mathbb{Z}) \cap H_{n-2,n-2}(X)$ is also surjective.

This part is relatively easy. Let $a \in H_{2(n-2)}(X) \cap H_{n-2,n-2}(X)$. Since $\sigma_* : H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z}) \rightarrow H_{2(n-2)}(X, \mathbb{Z})$ is surjective and $\sigma_* \otimes \mathbb{C} : H_{2(n-2)}(\tilde{X}_Y, \mathbb{C}) \rightarrow H_{2(n-2)}(X, \mathbb{C})$ preserves the Hodge type, there exists an element $b \in H_{2(n-2)}(\tilde{X}_Y, \mathbb{Z}) \cap H_{n-2,n-2}(\tilde{X}_Y)$ such that $\sigma_*(b) = a$. Now by assumption, we have an element $\tilde{b} \in L_{n-2}H_{2(n-2)}(\tilde{X}_Y)$ such that $\Phi_{n-2,2(n-2)}(\tilde{b}) = b$. Set $\tilde{a} \equiv \sigma_*(\tilde{b})$. Then from the commutative of the diagram, we have $\Phi_{n-2,2(n-2)}(\tilde{a}) = a$. This is exactly the surjectivity in this case.

This completes the proof for a blowup along a smooth codimension at least two subvariety Y in X . \square

2.2 The Proof of Theorem 1.2 for a Blowup

Now we have the following:

Proposition 2.1 *The assertion that “ $T_{n-2}H_k(X, \mathbb{Q}) = \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ holds” is a birationally invariant property of smooth n -dimensional varieties X when $k \geq 2(n-2)$.*

Proof. There are two cases to consider:

Case A: If $\Phi_{n-2,k} : L_{n-2}H_k(X) \otimes \mathbb{Q} \rightarrow \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ is surjective, we want to show $\Phi_{n-2,k} : L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q} \rightarrow \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$ is also surjective.

Let $a \in \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$, set $b = \sigma_*(a) \in \tilde{F}_{n-2}H_k(X, \mathbb{Q})$. By assumption, there exists $\tilde{b} \in L_{n-2}H_k(X, \mathbb{Q})$ such that $\Phi_{n-2,k}(\tilde{b}) = b$. By the blowup formula in Lawson homology (see [H1]), we know that $\sigma_* : L_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow L_{n-2}H_k(X, \mathbb{Q})$ is surjective, there exists an element $\tilde{a} \in L_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\sigma_*(\tilde{a}) = \tilde{b}$. By the commutative diagram in Lemma 2.1 and Corollary 2.1, we have $j^*(\Phi_{n-2,k}(\tilde{a}) - a) = 0 \in H_k^{BM}(U, \mathbb{Q})$. The exactness of the localization sequence in the rows in Corollary 2.1 implies that there exists an element $c \in H_k(D, \mathbb{Q})$ such that $i_*(c) = \Phi_{n-2,k}(\tilde{a}) - a$. Since the $\dim(D) = n-1$ and D is smooth, by Theorem 2.1, we know the natural transformation $\Phi_{n-2,k} : L_{n-2}H_k(D) \rightarrow H_k(D)$ is an isomorphism for $k \geq 2(n-2) + 1$. Hence $\Phi_{n-2,k} : L_{n-2}H_k(D) \otimes \mathbb{Q} \cong H_k(D, \mathbb{Q})$. Therefore there exists $\tilde{c} \in L_{n-2}H_k(D) \otimes \mathbb{Q}$ such that $\Phi_{n-2,k}(\tilde{c}) = c$. Now it is obvious that $\Phi_{n-2,k}(\tilde{a} - i_*(\tilde{c})) = a$. The proof of the case $k = 2(n-p)$ is from the proof of Theorem 1.1. This is the surjectivity as we want.

Case B: If $\Phi_{n-2,k} : L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q} \rightarrow \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$ is surjective, we want to show $\Phi_{n-2,k} : L_{n-2}H_k(X) \otimes \mathbb{Q} \rightarrow \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ is also surjective. We can use an argument similar to the **Case 2** above. Suppose $b \in \tilde{F}_{n-2}H_k(X, \mathbb{Q})$. Then there exists a $\tilde{b} \in \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\sigma_*(\tilde{b}) = b$ by the blowup formula for the singular homology with \mathbb{Q} -coefficients. By assumption, there exists an $\tilde{a} \in L_{n-2}H_k(\tilde{X}_Y) \otimes \mathbb{Q}$ such that $\Phi_{n-2,k}(\tilde{a}) = \tilde{b}$. Set $a = \sigma_*(\tilde{a})$. Then $a \in L_{n-2}H_k(X) \otimes \mathbb{Q}$ and $\Phi_{n-2,k}(a) = b$. This finishes the proof of the surjectivity in this case. \square

Now we give the proof of Theorem 1.2. First, we suppose that $G_{n-2}H_k(X, \mathbb{Q}) = \tilde{F}_{n-2}H_k(X, \mathbb{Q})$. We will show $G_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$ case by case.

For $k > 2n$, $\tilde{F}_{n-2}H_k(\tilde{X}_Y) = 0$ and hence nothing needs to be proved.

For $k = 2n$, $G_{n-2}H_k(\tilde{X}_Y) = \tilde{F}_{n-2}H_k(\tilde{X}_Y) = \mathbb{Z}$, so the result is true.

For $k = 2n - 1, 2n - 2$, $G_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) = H_k(\tilde{X}_Y, \mathbb{Q})$ follows from the definitions of the geometric filtration and the Hodge filtration.

The only case left is $k = 2n - 3$ since the case that $k = 2n - 4$ has been proved in Theorem 1.1. In this case, $T_{n-2}H_k(M, \mathbb{Q}) = G_{n-2}H_k(M, \mathbb{Q})$ has been proved in [H2] for any smooth projective variety M . The assumption $G_{n-2}H_k(X, \mathbb{Q}) = \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ is equivalent to $T_{n-2}H_k(X, \mathbb{Q}) = \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ in this situation. Hence $T_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$ follows from Proposition 2.1. Now by (2), we have $G_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$.

On the other hand, it has been proved in [[Lew1], Lemma 13.6] that $G_{n-2}H_k(X, \mathbb{Q}) \cong \tilde{F}_{n-2}H_k(X, \mathbb{Q})$ holds if $G_{n-2}H_k(\tilde{X}_Y, \mathbb{Q}) \cong \tilde{F}_{n-2}H_k(\tilde{X}_Y, \mathbb{Q})$. The last part is exactly the assumption. This completes the proof of Theorem 1.2 for one blowup over a smooth subvariety of codimension at least two. \square

2.3 The Proof of Theorem 1.3 for a Blowup

Similarly, for 1-cycles, we have the following.

Proposition 2.2 *For integer $k \geq 2$, the assertion that “ $T_1H_k(X, \mathbb{Q}) = \tilde{F}_1H_k(X, \mathbb{Q})$ holds” is a birationally invariant property of smooth n -dimensional varieties X .*

Proof. As before, there are two cases to consider:

Case a: If $T_1H_k(X, \mathbb{Q}) = \tilde{F}_1H_k(X, \mathbb{Q})$ holds, then $T_1H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_1H_k(\tilde{X}_Y, \mathbb{Q})$ holds. By the theorems in [[FM], §7], $T_1H_k(M, \mathbb{Q}) \subseteq \tilde{F}_1H_k(M, \mathbb{Q})$ holds for any smooth variety M . We only need to show $T_1H_k(\tilde{X}_Y, \mathbb{Q}) \supseteq \tilde{F}_1H_k(\tilde{X}_Y, \mathbb{Q})$. The argument is similar to the proof of the Theorem 1.3 in [H2]. I give the detail as follows:

Let $a \in \tilde{F}_1H_k(\tilde{X}_Y, \mathbb{Q})$, set $b = \sigma_*(a) \in \tilde{F}_1H_k(X, \mathbb{Q})$. By assumption, there exists $\tilde{b} \in L_1H_k(X, \mathbb{Q})$ such that $\Phi_{1,k}(\tilde{b}) = b$. By the blowup formula in Lawson homology (see [H1]), we know that $\sigma_* : L_1H_k(\tilde{X}_Y, \mathbb{Q}) \rightarrow L_1H_k(X, \mathbb{Q})$ is surjective, there exists an element $\tilde{a} \in L_1H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\sigma_*(\tilde{a}) = \tilde{b}$. By the commutative diagram in Lemma 2.1 and Corollary 2.1, we have $j^*(\Phi_{1,k}(\tilde{a}) - a) = 0 \in H_k^{BM}(U, \mathbb{Q})$. The exactness of the localization sequence in the rows in Corollary 2.1 implies that there exists an element

$c \in H_k(D, \mathbb{Q})$ such that $i_*(c) = \Phi_{1,k}(\tilde{a}) - a$. Set $d = \pi_*(c) \in L_1 H_k(Y) \otimes \mathbb{Q}$. By the commutative diagram in Corollary 2.1, d maps to zero under $(i_0)_* : H_k(Y, \mathbb{Q}) \rightarrow H_k(X, \mathbb{Q})$. Hence there exists an element $e \in H_{k+1}^{BM}(U, \mathbb{Q})$ such that $(\delta_0)_*(e) = d$. Let $\tilde{d} \in H_k(D, \mathbb{Q})$ be the image of e under this boundary map $\delta_* : H_{k+1}^{BM}(U, \mathbb{Q}) \rightarrow H_k(D, \mathbb{Q})$, i.e., $\tilde{d} = \delta_*(e)$. Therefore, the image of $c - \tilde{d}$ is zero in $H_k(Y, \mathbb{Q})$ under π_* and is zero in $H_k(\tilde{X}_Y, \mathbb{Q})$ under i_* . By the blowup formula in Lawson homology (see [H1]), we know such an element $c - \tilde{d}$ in the image of some $f \in L_1 H_k(D) \otimes \mathbb{Q}$, i.e., $\Phi_{1,k}(f) = c - \tilde{d}$. Hence we get $\Phi_{1,k}(\tilde{a} - i_*(f)) = a$. This is the surjectivity as we want.

Case b: If $T_1 H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q})$ holds, then $T_1 H_k(X, \mathbb{Q}) = \tilde{F}_1 H_k(X, \mathbb{Q})$ holds. This part is relatively easy. As before, we only need to show $T_1 H_k(X, \mathbb{Q}) \supseteq \tilde{F}_1 H_k(X, \mathbb{Q})$.

Let $b \in \tilde{F}_1 H_k(X, \mathbb{Q})$. Since $\sigma : \tilde{X}_Y \rightarrow X$ is the blowup along the smooth variety Y , we have $\sigma_*(\tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q})) \subseteq \tilde{F}_1 H_k(X, \mathbb{Q})$. In fact, the inclusion is an equality. (See [Lew2] Lemma.13.6) Therefore, there is an element $a \in \tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\sigma_*(a) = b$. By assumption, there is an element $\tilde{a} \in L_1 H_k(\tilde{X}_Y, \mathbb{Q})$ such that $\Phi_{1,k}(\tilde{a}) = a$. Set $\tilde{b} = \Phi_{1,k}(\tilde{a}) \in L_1 H_k(X, \mathbb{Q})$. By the naturality of $\Phi_{1,k}$, we have $\sigma_*(\tilde{b}) = b$. This is the surjectivity as we need. \square

Now we give the proof of Theorem 1.3. First, suppose $G_1 H_k(X, \mathbb{Q}) = \tilde{F}_1 H_k(X, \mathbb{Q})$. We want to show that $G_1 H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q})$.

Now comparing the blowup formula for Lawson homology (cf. [H1]) and for singular homology (both with \mathbb{Q} coefficients) along the same smooth subvariety Y of codimension at least two, we find the same new components, i.e.,

$$\bigoplus_{j=1}^{r-1} H_{k-2j}(Y, \mathbb{Q}),$$

both in $L_1 H_k(\tilde{X}_Y, \mathbb{Q})$ and $H_k(\tilde{X}_Y, \mathbb{Q})$.

This, together with (2), implies that the new component of this blowup along Y in $G_1 H_k(\tilde{X}_Y, \mathbb{Q})$ contains $\bigoplus_{j=1}^{r-1} H_{k-2j}(Y, \mathbb{Q})$. Since $G_1 H_k(\tilde{X}_Y, \mathbb{Q}) \subseteq H_k(\tilde{X}_Y, \mathbb{Q})$, the new component of this blowup along Y in $G_1 H_k(\tilde{X}_Y, \mathbb{Q})$ is also contained in $\bigoplus_{j=1}^{r-1} H_{k-2j}(Y, \mathbb{Q})$. Therefore

$$G_1 H_k(\tilde{X}_Y, \mathbb{Q}) \cong \left\{ \bigoplus_{j=1}^{r-1} H_{k-2j}(Y, \mathbb{Q}) \right\} \bigoplus G_1 H_k(X, \mathbb{Q}) \quad (3)$$

Similarly,

$$\tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q}) \cong \left\{ \bigoplus_{j=1}^{r-1} H_{k-2j}(Y, \mathbb{Q}) \right\} \bigoplus \tilde{F}_1 H_k(X, \mathbb{Q}) \quad (4)$$

From (3) and (4), we deduce that $G_1 H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q})$.

On the other hand, we also need to show that if $G_1 H_k(\tilde{X}_Y, \mathbb{Q}) = \tilde{F}_1 H_k(\tilde{X}_Y, \mathbb{Q})$, then $G_1 H_k(X, \mathbb{Q}) = \tilde{F}_1 H_k(X, \mathbb{Q})$. An argument similar to the one given in **Case B** works. Lewis [[Lew1], Lemma 13.6] proved this part in a more general setting.

This finishes the proof of Theorem 1.3 for a blowup along a smooth subvariety with codimension at least two. \square

Now recall the weak factorization Theorem proved in [AKMW] (and also [W]) as follows:

Theorem 2.2 ([AKMW] Theorem 0.1.1, [W]) *Let $f: X \rightarrow X'$ be a birational map of smooth complete varieties over an algebraically closed field of characteristic zero, which is an isomorphism over an open set U . Then f can be factored as*

$$X = X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n+1}} X_n = X'$$

where each X_i is a smooth complete variety, and $\varphi_{i+1}: X_i \rightarrow X_{i+1}$ is either a blowing-up or a blowing-down of a smooth subvariety disjoint from U .

Moreover, if $X - U$ and $X' - U$ are simple normal crossings divisors, then the same is true for each $X_i - U$, and the center of the blowing-up has normal crossings with each $X_i - U$.

Hence $\text{Hodge}^{2,2}(X, \mathbb{Q})$, $\widetilde{GHC}(n-2, k, X)$ and $\widetilde{GHC}(1, k, X)$ are birationally invariant properties about the smooth manifold X . \square

The proof of the Corollary 1.1 and 1.2 are based on Theorem 1.1, Remark 1.1 and the strong Lefschetz Theorem. By using the strong Lefschetz Theorem, one can show that $\text{Hodge}^{p,p}(X, \mathbb{Q}) \Rightarrow \text{Hodge}^{n-p, n-p}(X, \mathbb{Q})$ for $2p \leq n$. (See [Lew1] for the details.) \square

The Corollary 1.3 is obvious from Theorem 1.2 and Theorem 1.3.

□

3 A Remark on Generalizations

From the proof of the Theorem 1.1 and 1.2, we can draw the following conclusions:

- (a) Fix $n > 0$ and $0 \leq p \leq n$. If we have $\text{Hodge}^{i,i}(Y, \mathbb{Q})$ for all $i \leq p$ and all smooth projective variety Y , i.e., the Hodge conjecture is true for every smooth projective variety Y with $\dim(Y) = n$ and for algebraic cycles with codimension $\leq p$, then $\text{Hodge}^{p+1,p+1}(X, \mathbb{Q})$ is a birational invariant statement for every smooth projective X with $\dim(X) \leq n+2$. For example, if we have $\text{Hodge}^{2,2}(Y, \mathbb{Q})$ for all 4-folds Y , then $\text{Hodge}^{p,p}(X, \mathbb{Q})$ is a birational statement for any integer $0 \leq p \leq \dim(X)$ and smooth projective varieties X with $\dim(X) \leq 7$.

For the Generalized Hodge Conjecture, we have

- (b) Fix $n > 0$ and $0 \leq p \leq n$. If we have $\widetilde{GHC}(i, k, Y)$ for $i \leq p$, i.e., the Generalized Hodge Conjecture is true for every smooth projective Y with $\dim(Y) = n$ and for algebraic cycles with codimension $\leq p$, then $\widetilde{GHC}(m - p - 1, k, X)$ is a birational invariant statement for every smooth projective variety X with $\dim(X) = m \leq n + 2$.

Similarly,

- (c) Fix $n > 0$ and $0 \leq p \leq n$. If we have $\widetilde{GHC}'(i, k, Y)$ for $i \leq p$, i.e., the Generalized Hodge Conjecture is true for every smooth projective Y with $\dim(Y) = n$ and for algebraic cycles with dimension $\leq p$, then $\widetilde{GHC}(p + 1, k, X)$ is a birational invariant statement for every smooth projective variety X with $\dim(X) = m \leq n + 2$.

As a corollary of part (b) and (c), we have, for example, if we have $\widetilde{GHC}(1, 3, Y)$ for all 3-folds Y , then $\widetilde{GHC}(p, k, X)$ is a birational statement for X with $\dim(X) \leq 5$.

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